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Levy processes and their distributions in terms of independence

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1 Introduction

1.1 Algebraic probability spaces and probability distributions

In this article, $\mathcal{A}$ always denotes a unital $*$-algebra over $\mathbb{C}$, or sometimes a unital C^*-algebra if needed. φ denotes a state, that is, a linear functional from $\mathcal{A}$ to $\mathbb{C}$ satisfying $\varphi(X^*X) \geq 0$ and $\varphi(1) = 1$. An *algebraic probability space* is a pair $(\mathcal{A}, \varphi)$ of a $*$-algebra and a state. $X \in \mathcal{A}$ is called a random variable. The *probability distribution* μ_X of a self-adjoint random variable $X \in \mathcal{A}$ is defined by

$$\int_{\mathbb{R}} f(x) d\mu_X(x) = \varphi(f(X)) \text{ for all polynomials } f(x).$$

μ_X necessarily exists. Moreover, μ_X is unique if the moment problem for the sequence $\{\varphi(X^n)\}_{n\geq 0}$ is determinate. In particular, μ_X uniquely exists as a probability measure with a compact support if X is an element of a C^*-algebra.

1.2 Independence in probability theory

Independence is a fundamental concept in probability theory. We look at this concept in terms of non-commutative probability. Remarkably, independence is not unique in an algebraic probability space: for instance, free independence [30] is another possible independence. The usual one, which we call tensor independence, is the most basic.

Let $(\Omega, \mathcal{F}, P)$ be a probability space. Random variables $X, Y \in L^\infty(\Omega, \mathcal{F})$ are independent if and only if

$$E[X^m Y^n] = E[X^m]E[Y^n] \text{ for all } m, n \in \mathbb{N}.$$

We can prove this equivalence easily as follows. It is immediate that $E[P(X)Q(Y)] = E[P(X)]E[Q(Y)]$ for all polynomials P, Q. Weierstrass' polynomial approximation then implies that $E[f(X)g(Y)] = E[f(X)]E[g(Y)]$ for all $f, g \in C_b(\mathbb{R})$. It is well known that this is equivalent to the independence of X and Y.

The above formulation of independence is important when we try to extend tensor independence to non-commutative algebras. We note that σ-fields $\mathcal{F}_1, \mathcal{F}_2 \subset \mathcal{F}$ are independent if and only if X, Y are independent for all $X \in L^\infty(\Omega, \mathcal{F}_1)$ and $Y \in L^\infty(\Omega, \mathcal{F}_2)$. Therefore, it is enough to consider only bounded random variables in this sense.

The associativity of independence is an important property. Let X, Y be bounded and independent random variables. Then

$$E(X^pY^q) = E(X^p)E(Y^q).$$

Now we consider three random variables X, Y, Z. First we assume that X, Y are independent and moreover, $\{X, Y\}$ and Z are independent. The notation $\{X, Y\}$ means the σ-field generated by X and Y. Then

$$E((X^pY^q)Z^r) = E(X^pY^q)E(Z^r) = E(X^p)E(Y^q)E(Z^r).$$

Next we assume that X and $\{Y, Z\}$ are independent, and moreover, Y, Z are independent. Then

$$E(X^p(Y^qZ^r)) = E(X^p)E(Y^qZ^r) = E(X^p)E(Y^q)E(Z^r).$$

Therefore, these two results coincide. The above argument seems to be trivial, but is important when we generalize independence to non-commutative probability spaces.

A consequence of the associativity is that we only have to define independence for two random variables; independence for more than two random variables can be naturally defined via associativity.

1.3 Universal independence and natural independence

We define four independences in an algebraic probability space $(\mathcal{A}, \varphi)$. Each independence allows us to calculate joint moments of independent random variables and, moreover, satisfies the condition of associativity. It is known that independence satisfying nice conditions such as associativity is classified into five kinds [4, 22, 23, 28]. The fifth independence, called anti-monotone independence, is essentially the same as monotone independence in this article, and therefore it is omitted here.

Let $\{\mathcal{A}_i\}_{i=1}^{\infty} \subset \mathcal{A}$ be subalgebras containing the unit of $\mathcal{A}$.

Definition 1.1. (Tensor independence). $\{\mathcal{A}_i\}_{i=1}^{\infty}$ are said to be tensor independent if

$$\varphi(X_1 \cdots X_n) = \prod_j \varphi\Big(\prod_{X_i \in \mathcal{A}_j} X_i \Big).$$

Definition 1.2. (Free independence [30]). $\{\mathcal{A}_i\}_{i=1}^{\infty}$ are said to be free independent if

$$\varphi(X_1 \cdots X_n) = 0$$

holds whenever $\varphi(X_k) = 0$ $X_k \in \mathcal{A}_{i_k}$ for any k and $i_1 \neq \cdots \neq i_n$. The last symbol denotes that $i_j \neq i_{j+1}$ for any $1 \leq j \leq n-1$.

By contrast, the following two independences are meaningful only for subalgebras without containing the unit of $\mathcal{A}$. Therefore, we let $\{\mathcal{A}_i\}_{i=1}^{\infty} \subset \mathcal{A}$ be subalgebras which *do not* contain the unit of $\mathcal{A}$.

Definition 1.3. (Boolean independence [29]). $\{\mathcal{A}_i\}_{i=1}^{\infty}$ are said to be Boolean independent if

$$\varphi(X_1 \cdots X_n) = \prod_i \varphi(X_i).$$

for $X_k \in \mathcal{A}_{i_k}$, $i_1 \neq \cdots \neq i_n$.

Definition 1.4. (Monotone independence [20]). $\{\mathcal{A}_i\}_{i=1}^{\infty}$ are said to be monotone independent if

$$\varphi(X_1 \cdots X_n) = \varphi(X_j)\varphi(X_1 \cdots \check{X}_j \cdots X_n)$$

for $X_k \in \mathcal{A}_{i_k}$ and j satisfying $i_{j-1} < i_j > i_{j+1}$.

The above independences are called *natural independences*. Among them, tensor, free and Boolean independences are called *universal independences*. Universal independences satisfy a stronger condition of *commutativity* which will be explained later.

Remark 1.5. In the usual probability theory, a canonical realization of independence is known: random variables $X_1(\omega) := \omega_1, X_2(\omega) := \omega_2$ ($\omega = (\omega_1, \omega_2) \in \mathbb{R}^2$) are tensor independent in $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2), \mu_1 \times \mu_2)$. Any one of natural independences has a similar canonical construction by using the free product of algebras [20].

If we consider two or more states such as an algebraic probability space $(\mathcal{A}, \varphi_1, \psi, \cdots)$, other nontrivial independences appear [9, 15, 16]. Also in this setting, one can introduce many probabilistic concepts such as cumulants, central limit theorems, convolutions of probability measures, analogues for the Fourier transform and infinitely divisible distributions. These problems are currently studied by researchers: see [2, 9, 15, 19, 25] for instance.

We define three independences in two states.

Definition 1.6. (Conditionally free independence [8]). Let $\mathcal{A}_i$ be $*$-subalgebras of $\mathcal{A}$ containing the unit of $\mathcal{A}$. $\{\mathcal{A}_i\}_{i=1}^{\infty}$ is said to be conditionally (or c- for short) free independent if:

CF1 The equality

$$\varphi(X_1 \cdots X_n) = \prod_{i=1}^{n} \varphi(X_i) \tag{1.1}$$

holds whenever $\psi(X_k) = 0$, $X_k \in \mathcal{A}_{i_k}$ for all k and $i_1 \neq \cdots \neq i_n$.

CF2 $\{\mathcal{A}_i\}_{i=1}^{\infty}$ is a free independent family with respect to ψ.

Definition 1.7. (Conditionally monotone independence [15]) Let $(\mathcal{A}, \varphi, \psi)$ be an algebraic probability space. We consider sublagebras $\{\mathcal{A}_i\}_{i\in I}$, each of which does not contain the unit of $\mathcal{A}$. $\mathcal{A}_i$ are said to be c-monotone independent if the following properties are satisfied for all elements $X_i \in \mathcal{A}_{k_i}$ and indices $i_1, \cdots, i_n$, $n \geq 1$:

CM1 $\varphi(X_1 \cdots X_n) = \varphi(X_1)\varphi(X_2 \cdots X_n)$ whenever $i_1 > i_2$;

CM2 $\varphi(X_1 \cdots X_n) = \varphi(X_1 \cdots X_{n-1})\varphi(X_n)$ whenever $i_n > i_{n-1}$;

CM3 $\varphi(X_1 \cdots X_n) = (\varphi(X_j) - \psi(X_j))\varphi(X_1 \cdots X_{j-1})\varphi(X_{j+1} \cdots X_n) + \psi(X_j)\varphi(X_1 \cdots X_{j-1}X_{j+1} \cdots X_n)$ whenever j satisfies $i_{j-1} < i_j > i_{j+1}$ and $2 \leq j \leq n-1$;

CM4 $\mathcal{A}_i$ are monotone independent with respect to ψ.

For a tuple $(i_1, \cdots, i_n)$ of natural numbers with neighboring numbers different, we define the sets of bottoms and peaks. Let $B(i_1, \cdots, i_n)$ be the set of points k such that $i_{k-1} > i_k < i_{k+1}$ and $P(i_1, \cdots, i_n)$ the set of points k such that $i_{k-1} < i_k > i_{k+1}$. If $k = 1$ or n, one inequality is eliminated.

Definition 1.8. (Ordered free independence [16]) Let $\mathcal{A}_i$ be subalgebras of $\mathcal{A}$ containing the unit of $\mathcal{A}$. Then $\mathcal{A}_i$ are said to be ordered free independent if the following property holds for any $X_k \in \mathcal{A}_{i_k}$ and $(i_1, \cdots, i_n)$ with neighboring numbers different.

OF $\varphi(X_1 \cdots X_n) = 0$ and $\psi(X_1 \cdots X_n) = 0$ whenever $\varphi(X_k) = 0$ holds for $k \in P(i_1, \cdots, i_n)$ and $\psi(X_k) = 0$ holds for $k \in B(i_1, \cdots, i_n)$.

All the above independences, except for tensor independence, are unified by one independence in three states.

Definition 1.9. (Indented independence [16]) Let $(\mathcal{A}, \varphi, \psi, \theta)$ be an algebraic probability space equipped with three states. Let $\mathcal{A}_i$ be subalgebras of $\mathcal{A}$ containing the unit of $\mathcal{A}$. Then $\mathcal{A}_i$ are said to be indented independent if the following properties hold for any $X_k \in \mathcal{A}_{i_k}$ and tuple $(i_1, \cdots, i_n)$ with neighboring numbers different.

I1 $\mathcal{A}_i$ are ordered free independent with respect to (ψ, θ).

I2 $\varphi(X_1 \cdots X_n) = 0$ whenever $\varphi(X_1) = 0$, $\psi(X_k) = 0$ for $k \in P(i_1, \cdots, i_n) \backslash \{1\}$ and $\theta(X_k) = 0$ for $k \in B(i_1, \cdots, i_n) \backslash \{1\}$.

The concept of natural independence can be easily extended to algebraic probability spaces with two or three states. In such an extended sense, the above independences are *natural.* In particular, they are associative. However, there are no results on classification of natural independences in more than one states. This is partially because a special difficulty arises in more than one states. In one state, natural independence was classified into five ones by Muraki without the use of positivity of a state; a unital linear functional is enough to classify the five ones. By contrast, there are many natural independences in two or more states if the assumption of positivity is removed [17].

We mention how several independences are unified by indented independence; see [16] for details. First, using indented independence, one can understand the reasons why subalgebras $\mathcal{A}_i$ are assumed not to contain the unit of $\mathcal{A}$ in monotone, Boolean and c-monotone independences.

Second, the associative law of monotone independence had been proved differently from free independence. However, indented independence enables us to understand the associative laws of monotone and free independences at the same time.

Third, indented independence explains how monotone partitions appear from linearly ordered non-crossing partitions.

Thus, indented independence unifies free, monotone and Boolean ones. A remaining important question is if it is possible to unify also tensor independence in terms of natural independence in multi states.

Tensor, free, Boolean and c-free independences are commutative in the sense that random variables X and Y are independent if and only if Y and X are independent. This concept of mutual independence, however, is not valid for monotone, c-monotone, ordered free and indented independences: Y and X are not independent in generic cases even if X and Y are independent. This asymmetry arises, for instance, in the characterization of a monotone convolution; see Theorem 3.1. This asymmetry sometimes makes it difficult to analyze convolutions and cumulants. In spite of such a difficulty, there is still similarity between asymmetric independences and symmetric ones. Such examples are found in Theorems 4.1, 4.2, 4.4, 4.5, 5.2.

2 Central Limit Theorems

Since we have several kinds of independence, there are several central limit theorems (or CLTs for short). Given a concept of independence, a CLT is formulated as follows. If $X_1, X_2, X_3, \cdots \in \mathcal{A}$ are i.i.d. random variables satisfying $\varphi(X_1) = 0$, $\varphi(X_1^2) = 1$, then the normalized sum

$$Y_N := \frac{X_1 + \cdots + X_N}{\sqrt{N}}$$

is known to converge to a limit in the sense of weak convergence of probability distributions. In other words, there exists a probability measure μ such that

$$\mu_{Y_N} \to \mu \qquad (N \to \infty).$$

If the number of states is larger than one, a CLT is formulated as follows. What we consider is an algebraic probability space $(\mathcal{A}, \varphi, \psi, \theta, \cdots)$ equipped with states. Let X_i be self-adjoint random variables such that

(1) X_i are identically distributed, that is, for any n, the moments $\varphi(X_i^n)$, $\psi(X_i^n)$, $\theta(X_i^n)$, $\cdots$ do not depend on i.

(2) X_i are independent.

(3) X_i have zero means and finite variances: $\varphi(X_i) = 0$, $\psi(X_i) = 0$, $\theta(X_i) = 0$, $\cdots$, and $\varphi(X_i^2) = \alpha^2, \psi(X_i^2) = \beta^2, \theta(X_i^2) = \gamma^2, \cdots$.

We have not assumed that the variances are equal to one, since difference among the variances yields a variety of limit distributions. Then we consider limit distributions $\lambda, \mu, \nu, \cdots$ which respectively appear as the distributions of $\frac{X_1+\cdots+X_N}{\sqrt{N}}$ under the states $\varphi, \psi, \theta, \cdots$.

The limit distributions are shown in Table 1. Except for the tensor independence, all limit distributions are expressed in terms of the Kesten distributions. In the table, $\lambda = \frac{t-1}{2t-1}$ for

Table 1: The limit distributions in CLTs

Independence	Limit distribution
Tensor	$\frac{1}{\sqrt{2\pi}} \exp(-\frac{x^2}{2})dx$ on $\mathbb{R}$ (Gaussian)
Free	$\frac{\sqrt{4-x^2}}{2\pi}dx$ on $[-2,2]$ (Wigner's semicircle law)
Boolean	$\frac{1}{2}(\delta_1 + \delta_{-1})$ (Bernoulli's law)
Monotone	$\frac{1}{\pi\sqrt{2-x^2}}dx$ on $[-\sqrt{2}, \sqrt{2}]$ (Arcsine law)
C-free C-monotone Ordered free Indented	$\frac{t}{\pi} \cdot \frac{\sqrt{2s-x^2}}{(1-2t)x^2+2st^2} 1_{\lvert x\rvert \le \sqrt{2s}}(x)dx + \lambda(\delta_{-a} + \delta_a)$ (Kesten distribution)

$t > 1$ and $\lambda = 0$ for $0 \le t \le 1$. a is defined by $a = t\sqrt{\frac{2s}{2t-1}}$. The parameters s, t are expressed in terms of $\alpha^2, \beta^2, \gamma^2$: $(s,t) = (\beta^2+\gamma^2, \frac{\alpha^2}{\beta^2+\gamma^2})$ for indented independence; $(s,t) = (2\beta^2, \frac{\alpha^2}{2\beta^2})$ for c-free independence; $(s,t) = (\alpha^2+\beta^2, \frac{\alpha^2}{\alpha^2+\beta^2})$ for ordered independence; $(s,t) = (\beta^2, \frac{\alpha^2}{\beta^2})$ for c-monotone independence.

Wigner's semicircle law, arcsine law and Bernoulli's law are all special cases of the Kesten distributions. This is a natural consequence of the fact that indented independence unifies free, monotone and Boolean independences.

3 Convolutions of probability distributions

Let X, Y be self-adjoint elements of a C^*-algebra and be independent in some sense. The convolution of μ_X and μ_Y is defined by μ_{X+Y} and is denoted as $\mu_X \star \mu_Y$. Depending on a choice of independence, $\star$ is denoted as $*$ for tensor independence, $\boxplus$ for free independence, $\triangleright$ for monotone independence and $\uplus$ for Boolean independence.

The tensor convolution is characterized by the multiplication of the Fourier transforms. The other three convolutions also have analogous characterizations. However, these three convolutions sharply differ from the tensor one since they are characterized by the Stieltjes transform, not by the Fourier transform.

We define the Stieltjes transform $G_\mu(z) := \sum_{n=0}^{\infty} \frac{m_n(\mu)}{z^{n+1}} = \int_{\mathbb{R}} \frac{1}{z-x} d\mu(x)$ for $z \notin \mathbb{R}$ and the Fourier transform $\mathcal{F}_\mu(z) := \int_{\mathbb{R}} e^{izx}\mu(dx)$, $z \in \mathbb{R}$. $F_\mu(z) := \frac{1}{G_\mu(z)}$ is called the reciprocal Cauchy transform of μ. $\phi_\mu(z) := F_\mu^{-1}(z) - z$ is defined in an open set $\Omega_\mu \subset \mathbb{C}$ and is called the Voiculescu transform [7]. We note that $\phi_\mu(\frac{1}{z})$ and sometimes $z\phi_\mu(\frac{1}{z})$ are called the R-transform of μ.

Theorem 3.1. *(1)* $\mathcal{F}_{\mu*\nu}(z) = \mathcal{F}_\mu(z)\mathcal{F}_\nu(z)$, $z \in \mathbb{R}$.
(2) (Bercovici-Voiculescu [7]) $\phi_{\mu\boxplus\nu}(z) = \phi_\mu(z) + \phi_\nu(z)$, $z \in \Omega_\mu \cup \Omega_\nu$.
(3) (Speicher-Woroudi [29]) $F_{\mu\uplus\nu}(z) = F_\mu(z) + F_\nu(z) - z$, $z \notin \mathbb{R}$.
(4) (Muraki [20]) $F_{\mu\triangleright\nu}(z) = F_\mu(F_\nu(z))$ *for* $z \notin \mathbb{R}$.

If we take the logarithm of the Fourier transforms, the tensor convolution is characterized by the sum of such transforms. In this sense, only monotone convolution is different from the other three. Still there exists a similar transform which is a vector field A_μ defined in an open set U_μ of $\mathbb{C}$ such that the flow $F_t(z)$ generated by A_μ satisfies $F_1 = F_\mu$. The existence of such a vector field is proved by using the uniformization theorem for a simply connected Riemannian surface. The reader is referred to [10] for the definition. In generic cases, $A_{\mu\triangleright\nu} \neq A_\mu + A_\nu$; however, this transform behaves additively for powers of a probability measure: $A_{\mu^{\triangleright n}}(z) = nA_\mu(z)$. This property is also observed in monotone cumulants [12].

4 Infinitely divisible distributions

μ is said to be $\star$-infinitely divisible if for any n, there exists a probability measure μ_n such that $\mu = \mu_n^{\star n}$. Infinitely divisible distributions appear as the probability distributions of Lévy processes. We however focus only on probability distributions, not on processes in this article. Accordingly to the four kinds of convolutions, there are four concepts of infinitely divisible distributions. The following theorem is classical and well known in probability theory [27].

Theorem 4.1. *The following are equivalent.*
(1) μ is $*$-infinitely divisible.
(2) There exist $\gamma \in \mathbb{R}$ and a non-negative finite measure τ such that

$$\mathcal{F}_\mu(z) = \exp\Big(i\gamma z + \int_{\mathbb{R}} (e^{izx} - 1 - \frac{ixz}{1+x^2})\frac{1+x^2}{x^2}\tau(dx)\Big).$$

(3) There exists a weakly continuous $*$-convolution semigroup $\{\mu_t\}_{t\geq 0}$ such that $\mu_0 = \delta_0$ and $\mu_1 = \mu$.

The representation in (2) is called the Lévy-Khintchine formula.

Analogous results are known for free [7] and monotone convolutions [1, 20].

Theorem 4.2. *The following are equivalent.*
(1) μ is $\boxplus$-infinitely divisible.
(2) There exist $\gamma \in \mathbb{R}$ and a non-negative finite measure τ such that

$$\phi_\mu(z) = \gamma + \int_{\mathbb{R}} \frac{1+xz}{z-x}\tau(dx).$$

(3) There exists a weakly continuous $\boxplus$-convolution semigroup $\{\mu_t\}_{t\geq 0}$ such that $\mu_0 = \delta_0$ and $\mu_1 = \mu$.

Theorem 4.3. *The following are equivalent.*
(1) μ is $\triangleright$-infinitely divisible.
(2) There exists a vector field A_μ of such a form as

$$A_\mu(z) = -\gamma + \int_{\mathbb{R}} \frac{1+xz}{x-z}\tau(dx),$$

where $\gamma \in \mathbb{R}$ and τ is a non-negative finite measure, and F_μ coincides with $\exp(A_\mu)$. $\exp(A_\mu)$ denotes the time one mapping F_1 of a flow $\{F_t\}_{t\geq 0}$ generated from the differential equation $\frac{d}{dt}F_t(z) = A_\mu(F_t(z))$, $F_0(z) = z$.
(3) There exists a weakly continuous $\triangleright$-convolution semigroup $\{\mu_t\}_{t\geq 0}$ such that $\mu_0 = \delta_0$ and $\mu_1 = \mu$.

Examples are shown in Table 2-4.

For the Boolean convolution, any probability measure is $\uplus$-infinitely divisible. The Lévy-Khintchine formula exists for any probability measure in the form

$$F_\mu(z) - z = -\gamma + \int_{\mathbb{R}} \frac{1+xz}{x-z}\tau(dx).$$

The probability distribution of an increasing Lévy process is intensively studied in probability theory. Such a distribution is important in the theory of subordination, that is, a random time change of a Lévy process. A basic example is a Poisson distribution. Such a probability distribution is characterized as follows. The reader is referred to [27] for the proof.

Theorem 4.4. *Let $\{\mu_t\}_{t\geq 0}$ be a weakly continuous $*$-convolution semigroup with $\mu_0 = \delta_0$. Then the following statements are equivalent:*

(1) there exists $t > 0$ such that supp $\mu_t \subset [0, \infty)$;

Table 2: Tensor infinitely divisible distributions.

	Probability measure	Support	γ	τ
Gauss	$\frac{1}{\sqrt{2\pi\sigma^2}}\exp(-\frac{(x-a)^2}{2\sigma^2})dx$	$\mathbb{R}$	a	$\sigma^2\delta_0$
Poisson	$e^{-\lambda}\sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\delta_n$	$\{0,1,2,\cdots\}$	$\frac{\lambda}{2}$	$\frac{\lambda}{2}\delta_1$
Delta	δ_a	$\{a\}$	a	0
Cauchy	$\frac{1}{\pi}\frac{b}{(x-a)^2+b^2}dx$	$\mathbb{R}$	a	$\frac{b}{\pi(x^2+1)}$ on $\mathbb{R}$
Gamma	$\frac{\alpha^c}{\Gamma(c)}x^{c-1}e^{-\alpha x}dx$	$[0,\infty)$	$c\int_0^\infty \frac{e^{-\alpha x}}{1+x^2}dx$	$\frac{x}{1+x^2}e^{-\alpha x}$ on $[0,\infty)$

Table 3: Free infinitely divisible distributions.

	Probability measure	Support	γ	τ
Wigner's semicircle	$\frac{\sqrt{4\sigma^2-(x-a)^2}}{2\pi\sigma^2}dx$	$[a-2\sigma, a+2\sigma]$	a	$\sigma^2\delta_0$
Marchenko-Pastur ($\lambda = 1$)	$\frac{1}{2\pi}\sqrt{\frac{4-x}{x}}dx$	$[0,4]$	$\frac{1}{2}$	$\frac{1}{2}\delta_1$
Delta	δ_a	$\{a\}$	a	0
Cauchy	$\frac{1}{\pi}\frac{b}{(x-a)^2+b^2}dx$	$\mathbb{R}$	a	$\frac{b}{\pi(x^2+1)}$ on $\mathbb{R}$

The Marchenko-Pastur law is the Poisson distribution in free probability.

Table 4: Monotone infinitely divisible distributions.

	Probability measure	Support	γ	τ
Centered arcsine	$\frac{1}{\pi\sqrt{2\sigma^2-x^2}}dx$	$[-\sqrt{2}\sigma, \sqrt{2}\sigma]$	0	$\sigma^2\delta_0$
Monotone Poisson	$\frac{1}{\pi}\mathrm{Im}\frac{1}{E^{-1}(e^{-1-\lambda})}dx + e^{-\lambda}\delta_0$	$\{0\}\cup[a,b]$	$\frac{\lambda}{2}$	$\frac{\lambda}{2}\delta_1$
Delta	δ_a	$\{a\}$	a	0
Cauchy	$\frac{1}{\pi}\frac{b}{(x-a)^2+b^2}dx$	$\mathbb{R}$	a	$\frac{b}{\pi(x^2+1)}$ on $\mathbb{R}$

E^{-1} is an inverse function of $z \mapsto ze^z$ and a, b are functions of λ. See [21] for details.

(2) supp $\mu_t \subset [0,\infty)$ *for any* $t > 0$*;*

(3) supp $\tau \subset [0,\infty)$, $\tau(\{0\}) = 0$, $\int_0^\infty \frac{1}{x}d\tau(x) < \infty$ *and* $\gamma \geq \int_0^\infty \frac{1}{x}d\tau(x)$.

There are analogues of the above result for monotone and Boolean convolution: the result for the monotone convolution was proved in [14] and for Boolean convolution in [3].

Theorem 4.5. *Let* $\{\mu_t\}_{t\geq 0}$ *be a weakly continuous* $\triangleright$ *(resp.* $\uplus$*)-convolution semigroup with* $\mu_0 = \delta_0$. *Then the following statements are equivalent:*

(1) there exists $t > 0$ *such that supp* $\mu_t \subset [0,\infty)$*;*

(2) supp $\mu_t \subset [0,\infty)$ *for any* $t > 0$*;*

(3) supp $\tau \subset [0,\infty)$, $\tau(\{0\}) = 0$, $\int_0^\infty \frac{1}{x}d\tau(x) < \infty$ *and* $\gamma \geq \int_0^\infty \frac{1}{x}d\tau(x)$.

The above theorem is not true for $\boxplus$-convolution semigroups. However, (2) and (3) are still equivalent also in free probability [5]. Probability measures satisfying the mutually equivalent conditions (2) and (3) are said to be *regular* [26]. Thus, among the four independences, only free probability shows an exceptional property of probability measures on $[0,\infty)$.

5 Convergence of probability measures to Cauchy distributions

In probability theory, stable distributions are well investigated. They can be defined at least in two ways [11, 27]: the first one is in terms of self-similarity of a Lévy process; the second is in terms of domains of attraction. There are also analogues for free, Boolean and monotone independences. The aspect of self-similarity is found in [7, 13, 29] and the aspect of domains of attraction is in [6, 18].

For Boolean independence, every stable distribution is strictly stable. The property has not been proved for monotone independence. These situations are due to the fact that Boolean and monotone independences for subalgebras become trivial if the subalgebras contain the unit of the whole algebra. As a consequence, $\delta_a \uplus \mu$ and $\delta_a \rhd \mu$ differ from the shifted measure $\delta_a * \mu$. For this reason, we will define domains of attraction for Boolean and monotone convolutions in a slightly different way.

From now on, let us consider only Cauchy distributions which are in particular important in tensor, free, Boolean and monotone independences. This is because they are strictly 1-stable distributions in the four independences. Let

$$\mu_{a,b}(dx) = \frac{1}{\pi}\cdot\frac{b}{(x-a)^2+b^2}dx$$

be the Cauchy distribution with parameters $a \in \mathbb{R}$ and $b \geq 0$. $\mu_{a,0}$ is defined to be δ_a. A probability measure μ is said to belong to the domain of attraction of the Cauchy distribution $\mu_{a,b}$ if there exist $a_n \in \mathbb{R}, b_n > 0$ such that for i.i.d. random variables X_n with distribution μ, the random variables

$$\frac{X_1+\cdots+X_n}{b_n} - a_n$$

converge to $\mu_{a,b}$ in distribution. These definitions are valid for tensor and free convolutions. For monotone and Boolean convolutions, this definition causes a problem since the constant a_n is not independent of X_i's in generic cases. Therefore, we also require $a_n = 0$ for monotone and Boolean convolutions.

Thus we have four kinds of domains of attractions accordingly to tensor, free, Boolean and monotone independences. Theorem 4.1 of the paper [6] implies the following result as a special case.

Theorem 5.1. *The domain of attraction of $\mu_{a,b}$ for the free convolution coincides with that for the tensor convolution.*

This is a consequence of the fact that $\mu_{a,b}$ is fixed by the Bercovici-Pata bijection [6].

In a paper [18], we proved the following result for the monotone convolution.

Theorem 5.2. *μ belongs to the $\rhd$-domain of attraction of $\mu_{a,b}$ if:*

(1) there exists $R > 0$ such that $\mu|_{|x|\geq R}$ has a density of the form $\sum_{n=2}^{\infty}\frac{a_n}{x^n}$ which absolutely converges for $|x| \geq R$;

(2) the first complex moment of μ is equal to $a + ib$.

The nth complex moment of μ is defined as the coefficient of $\frac{1}{z^{n+1}}$ in the power expansion of $G_\mu(z)$, Im $z < 0$. The reader is referred to [18] for details.

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